REPORTS@SCM

AN ELECTRONIC JOURNAL OF THE
SOCIETAT CATALANA DE MATEMÀTIQUES

A random walk approach to Stochastic Calculus

*Salim Boukfal Lazaar
Universitat Autònoma de Barcelona (UAB)
salim.boukfal.lazaar@gmail.com

*Corresponding author

Resum *(CAT)*
L'objectiu d'aquest treball és presentar una introducció al càlcul estocàstic. En la primera part parlem del moviment brownià, el qual veurem que es pot pensar com a límit de passeigs aleatoris amb l'ajut del principi d'invariància de Donsker.
A continuació, presentem de manera heurística les equacions diferencials estocàstiques i veiem com es poden definir de manera rigorosa amb l'ajut de la integral estocàstica. Finalment, parlem d'existència i unicitat de solucions d'aquestes equacions i tractem un cas senzill com és el de l'equació de Langevin.

Abstract *(ENG)*
The aim of this work is to provide an introduction to the subject of Stochastic Calculus. In the first part we talk about the Brownian motion, which we will see that it can be thought as a limit of random walks via Donsker's Invariance Principle. Next, we heuristically present the stochastic differential equations and see how they can be rigorously defined with the help of the stochastic integral. Finally, we discuss the matter of existence and uniqueness of solutions to such equations and solve a rather simple case like the Langevin equation.

Societat Catalana de Matemàtiques

Institut d'Estudis Catalans

Keywords: *Brownian motion, random walk, stochastic integral, stochastic differential equation, Langevin.*
MSC (2020): *60B10, 60F17, 60G50, 60H05, 60H10, 60J65.*

Received: *June 23, 2024.*
Accepted: *September 6, 2024.*

1. Introduction

In subjects like Thermodynamics and Statistical Mechanics, in several occasions one gives a stochastic approach of a problem even though it can be treated in a deterministic way because it usually leads to simple and less tedious formulations and computations. For instance, if we want to study the motion of a particle in a fluid, it is much more simpler to think that the object moves randomly due to the several collisions that are happening in the system, rather than considering each interaction individually and try to force brute Newton's equations into the system.

This approach, which seems promising, comes with a couple of drawbacks. The first one is that we have to give up on trying to determine the exact trajectory of the particle, since, even if the initial conditions are the same, different identical particles might describe different sample paths.

The other drawback, which we shall focus our attention on, is that these kind of formulations usually lead to equations like the Langevin equation:

$$\frac{d\dot{X}_t}{dt} = -\mu \dot{X}_t + \dot{F}_t, \quad \dot{X}_t = \frac{dX_t}{dt}, \tag{1}$$

where μ is some positive real constant, X_t is the position of the particle at time $t \geq 0$ and $\dot{F}_t$ is a random perturbation that evolves with time and satisfies some conditions like $\mathbb{E}[\dot{F}_t] = 0$ and $\mathbb{E}[\dot{F}_t \dot{F}_s] = \Gamma \delta(t-s)$ (being $\mathbb{E}$ the expectation operator, Γ some positive real constant and δ the Dirac delta). Many physicists say that the process $X = \{X_t : t \geq 0\}$, where X_t is the position described by the latter equation, is a Brownian motion. However, it is very well-known that the sample paths of such process are nowhere differentiable in closed intervals with probability one, meaning that expressions like $\dot{X}_t$ (and higher order derivatives) make no sense when they are considered pathwise, so we must find a way to define such objects (derivatives of functions which are not differentiable in the usual sense) in order to be able to give a rigorous definition of equations like (1). Before doing so, we first need to define what is a Brownian motion. More particularly, we must check that we can define a mathematical object satisfying the properties that a process like the one described by (1) should satisfy.

2. Construction of the Brownian motion

When one asks what is a Brownian motion to someone who is not familiar with the subject of stochastic processes, the usual answer is that it is the random movement of a particle suspended in some medium (a liquid or the air, for instance). In some other cases, the answer is that it is the movement described by a particle that makes small, random displacements which behave similarly, even though they seem uncorrelated no matter what the position of the object is.

But all these features are already satisfied by a random walk whose jumps are "small" (for instance, of finite variance). Indeed, recall that a random walk is a process $S = \{S_t : t \in \mathbb{N} \cup \{0\}\}$ such that $S_0 = 0$ (this is taken arbitrarily) and

$$S_t = \sum_{j=1}^{t} X_j, \quad t \geq 1,$$

where $\{X_j : j \in \mathbb{N}\}$ is a sequence of i.i.d. random variables, which we shall assume, without any loss of generality, that they are centered and with variance $0 < \sigma^2 < \infty$. So why would we need to give it another

name? What is the difference between this processes and the so-called Brownian motion? To see this, we first see some of the common properties that share the class of random walks with finite variance jumps:

1. The first one, which is a choice rather than some intrinsic property of the process S, is that it starts from the origin.

2. The second one, which is a bit more interesting, is that the displacements of the process are independent and stationary; that is, if $0 \leq s < t \leq s' < t'$, then the random variables $S_t - S_s$ and $S_{t'} - S_{s'}$ are independent and the law of $S_t - S_s$ depends only on $t - s$. Indeed, for the independence of the increments, one has that

$$S_t - S_s = \sum_{j=s+1}^{t} X_j, \quad S_{t'} - S_{s'} = \sum_{j=s'+1}^{t'} X_j.$$

   Since the random variables X_j are mutually independent, we conclude that the increments are independent. As for the second part, the fact that the law of the increment $S_t - S_s$ depends only on $t - s$ means, in our setting, that the law depends only on the number of variables X_j involved. Since they are independent and identically distributed, the claim follows.

3. The last property, but not less important, is that, due to the Central Limit Theorem, for $t \geq 0$ large enough, and roughly speaking,

$$S_t \sim \mathcal{N}(0, \sigma^2 t).$$

   In other words, the long term behaviour of the random variable S_t is described by a centered Gaussian random variable with variance $\sigma^2 t$. Since it depends linearly with time, one can say that the process is diffusive in the long term.

Therefore, it seems that, when the right scales are considered, all random walks behave in the same way (modulo some constant). This is the content of Donsker's Invariance Principle (Theorem 2.2), which we state below. Before doing so, we must first define mathematically what a Brownian motion is.

Definition 2.1. A stochastic process $B = \{B_t : t \in \mathbb{R}_+\}$ is a one-dimensional Brownian motion if:

1. $B_0 = 0$ almost surely.

2. For any $k \in \mathbb{N}$ and any $0 \leq t_1 < \cdots < t_k < \infty$, the random variables $B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_k} - B_{t_{k-1}}$ are independent.

3. For any $0 \leq s < t < \infty$, the random variable $B_t - B_s$ is normally distributed with zero mean and variance $\sigma^2(t - s)$ for some constant $0 < \sigma < \infty$.

4. The sample paths of the process are continuous everywhere with probability one.

The process B is said to be a standard Brownian motion if $\sigma = 1$.

Observe that many of the properties of the random walk are shared by the Brownian motion. An additional property has been added, which is that the sample paths of the process are continuous with probability one, however, this is not so important, since, if the first three properties are satisfied, one can find a version of the process satisfying the fourth one.

In the case of the random walk, we provided a class of processes (which were determined by the sequence of random variables $\{X_j : j \in \mathbb{N}\}$) that satisfied the first three properties. However, the same cannot be done in the case of the Brownian motion, which can be thought as a continuous time version of the random walk. Hence, we have to first check that such process exists. This is, as well, part of the content of Donsker's Theorem, which we now state.

Theorem 2.2 (Donsker's Invariance Principle)**.** *Let $\{X_j : j \in \mathbb{N}\}$ be a sequence of independent and identically distributed centered random variables with unitary variance. Then the random (continuous) functions*

$$Y_t^{(n)} = \frac{1}{\sigma\sqrt{n}}\tilde{S}_{nt}, \quad 0 \leq t \leq 1,$$

where

$$\tilde{S}_t = \sum_{j=1}^{[t]} X_j + (t - [t])X_{[t]+1}, \quad \tilde{S}_0 = 0,$$

converge weakly to a standard one-dimensional Brownian motion, where $[t]$ denotes the integer part of t. In other words, if P_n are the laws of the random functions $Y_t^{(n)}$, then there is a probability measure P (the Wiener measure) over the space of real continuous functions on $[0,1]$, $C[0,1]$, fulfilling the properties from Definition 2.1 and such that $P_n(G) \to P(G)$ for any Borel set G of $C[0,1]$ with $P(\partial G) = 0$, being ∂G the boundary of G.

The process $\tilde{S}$, which resembles quite a lot S, is the linear interpolation of the latter and hence, a process with continuous sample paths.

The proof of this result (which is a result of convergence of probability measures), relies, mainly, on Prohorov's Theorem, which gives a characterization of the family of laws induced by the family of random functions $\{Y^{(n)} : n \in \mathbb{N}\}$, with $Y^{(n)} = \{Y_t^{(n)} : t \in [0,1]\}$ in terms of the topological properties of the space $C[0,1]$, and the fact that the finite dimensional distributions of a continuous stochastic process determine its law (we refer to [1, Theorems 5.1, 5.2 and p. 84] for a proof of these claims). A proof of Theorem 2.2 for a particular case of random walk is given in [2], and a general proof can be found in [1, Section 8] as well.

With this, we have given an answer to the first of the two questions and now can address the problem of defining objects like (1).

3. Stochastic differential equations

Before trying to define the concept of solution to equations like (1), which are known as stochastic differential equations (SDEs), we shall first see how one gets to the point of having to consider such objects.

To do so, let us consider an ordinary differential equation (ODE) of the form

$$dX_t = f(t, X_t)\, dt, \quad t \geq 0, \tag{2}$$

modeling some phenomena which we are interested in and where $f\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is some good enough function.

In some cases, the description given by the ODE might be a bit too simple or might not take into account some factors which might have been neglected due to a simplification or due to the fact that we cannot easily control them. To solve this, one can discretize the ODE and add a random perturbation which might evolve with time, say $V = \{V_t : t \in \mathbb{R}_+\}$, leading to

$$X_{t+\Delta t} - X_t = f(t, X_t)\Delta t + g(t, X_t)\Delta V_t, \quad \Delta V_t = V_{t+\Delta t} - V_t,$$

where $g\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is some function modelling the intensity of the random perturbation. Usually, this introduced noise accounts for the superposition of several small (of finite variance) factors which cannot be controlled. Hence, and due to the Central Limit Theorem, we can assume that the law of the increments ΔV_t is normally distributed with vanishing mean (since the mean trajectories should coincide with the one modeled by (2)) and with variance Δt. The linear dependence on time in the variance is chosen because, in most scenarios, the observed perturbation can be said to be diffusive.

One can assume, as well, that the random perturbations in discrete time, $\Delta V_0, \Delta V_{\Delta t}, \dots$ are uncorrelated or independent since they are supposed to be rapidly varying and hence, what happens in one time interval might not significantly interfere on what happens in some other time interval.

With all this, one concludes that the best choice for the process V is a standard Brownian motion. The only think left to do is to take the limit $\Delta t \to 0$ to obtain, formally speaking,

$$dX_t = f(t, X_t)\, dt + g(t, X_t)\, dB_t.$$

However, and as mentioned in the introduction, the differential dB_t makes no sense as a classical one. To solve this problem, one writes the SDE in its integral form

$$X_t = X_0 + \int_0^t f(s, X_s)\, ds + \int_0^t g(s, X_s)\, dB_s. \tag{3}$$

So the only thing left to do is to give a meaning to expressions $\int_0^t X_s\, dB_s$ (stochastic integral) for a suitable class of stochastic processes $X = \{X_t : t \in \mathbb{R}_+\}$ to solve the problem.

3.1 Stochastic integrals

The first idea to approach such integrals is to use the already developed theory of integration with respect to functions (Lebesgue–Stieltjes integral) to define such integrals pathwise. However, the fact that the sample paths of the Brownian motion are of unbounded variation preclude this option.

For this purpose, a new theory of integration needs to be developed. As in the case of the Riemann–Stieltjes integral, we will be considering sums of the form

$$\sum_{j=0}^{n-1} X_{t_j^*}(B_{t_{j+1}} - B_{t_j}), \tag{4}$$

where $0 = t_0 < \cdots < t_n = T$ is a partition of a finite time interval $[0, T]$ and where $t_j^* \in [t_j, t_{j+1})$, $j = 0, \dots, n-1$. Ideally, one would want the above sums to converge to the same limit (this limit might be in probability or in mean square, for instance) no matter what choice of t_j^* is made. Unfortunately, this is not the case, leading to different definitions of the stochastic integral depending on the choice of the midpoints t_j^*, $j = 0, \dots, n-1$. In this work, we will be considering the left endpoint approximations ($t_j^* = t_j$), which lead to the Itô integral.

As one might expect, this integral will not be defined for any process X. Returning to the discretization of the SDE, we have that the information we have on the process X at time $t+\Delta t$ can be determined by the information we have on X_t and the information we have on the driving noise (in our case, the Brownian motion) at time $t+\Delta t$. At the same time, the information we have of X_t depends on the information one has on $X_{t-\Delta t}$ and so on. All in all, we see that we can infer the information of X_t at time t by knowing the entire information of the driving process B until that time. In particular, the information we have on X_t does not depend on the information we have at time s for $s > t$, so the process X cannot see into the future. In this case, we say that the process X must be adapted to the filtration generated by the driving noise (the information we have on X_t depends on the history of the noise until that time).

Another natural hypothesis on the process X is that it must be integrable in some sense so that we can talk about its integral. More precisely, we will require that

$$||X||^2 := \mathbb{E}\left[\int_0^T X_s^2\, ds\right] < \infty.$$

For this integral to be well defined, we will require, as well, the process X, thought as a map $X\colon \Omega\times[0,\,T] \to \mathbb{R}$, $(\omega,\,t) \mapsto X(\omega,\,t) = X_t(\omega)$, where Ω is the sample space, to be jointly measurable with respect to the corresponding σ-fields.

When all these hypothesis are fulfilled, one can show that integrals like $\int_0^t X_s\, dB_s$ can be defined as an $L^2(\Omega)$-limit (mean square limit) of Riemann–Stieltjes sums. To show this, and as it is customary in this type of constructions, one first defines a class of simple functions of the form

$$\phi(\omega,\,t) = \phi_t(\omega) = \sum_{j=0}^{n-1} e_j(\omega)\mathbb{I}_{[t_j,\,t_{j+1})}(t), \tag{5}$$

where $\{e_j : j = 0,\,\dots,\,n-1\}$ are bounded random variables such that the information we have on e_j depends only on the history of the Brownian motion (the driving noise) until time t_j and $0 = t_0 < \cdots < t_n = T$. For such functions, the integral with respect to the Brownian motion is defined as the sum (4), where $X_{t_j^*}$ must be replaced by e_j.

Next, one checks that $||\cdot||$ defines a norm on the space of processes X satisfying the previously mentioned hypothesis and that such normed space (from now on, the space of Itô integrable processes) is complete.

Finally, one shows that any process X in the normed space can be approximated by simple functions (5), which allows us to define the integral $\int_0^T X_s\, dB_s$ as an $L^2(\Omega)$-limit of integrals of simple processes. To justify this last step, a crucial result for step functions (which also holds for general Itô integrable processes X) is needed. We shall state the result, as it will be useful in the future for other purposes.

Theorem 3.1 (Isometry formula)**.** *For any Itô integrable process X, we have*

$$\mathbb{E}\left[\left(\int_0^T X_s\, dB_s\right)^2\right] = \mathbb{E}\left[\int_0^T X_s^2\, ds\right].$$

As its name says, the previous result asserts that the stochastic integral with respect to the Brownian motion establishes an isometry between the space of square integrable random variables, $L^2(\Omega)$, and the space of Itô integrable functions. For a detailed construction of the stochastic integral, we refer to Chapter 3 of [3] and [4].

Another important feature of this integral is that, when X is a deterministic Itô integrable process (that is, the map $X : \Omega \times [0, T]$ is constant in the first argument), one has that the process $I = \{I_t : t \in [0, T]\}$ defined by $I_t = \int_0^t X_s \, dB_s$ is a Gaussian process. More precisely,

Theorem 3.2. *If $X = f = \{f_t : t \in [0, T]\}$ is a deterministic Itô integrable process, then I is a centered Gaussian process with independent increments such that, for each $0 \leq s < t \leq T$,*

$$I_t - I_s = \int_s^t f_s \, dB_s \sim \mathcal{N}\left(0, \int_s^t f_u^2 \, du\right).$$

That is, the increment is normally distributed with zero mean and variance $\int_s^t f_u^2 \, du$.

With this, the task of giving a meaning to expressions like (3) has been fulfilled. However, we have not provided any practical way of computing stochastic integrals. This will be the purpose of the Itô formula (see [3, Chapter 3] again or [4, Chapter 4] for a proof of this result), which can be thought as a chain rule or as an integration by parts formula, depending on whether you consider the differential or integral form.

Theorem 3.3 (Itô formula)**.** *Let $X = \{X_t t \in [0, T]\}$ be a process defined by*

$$dX_t = f_t \, dt + g_t \, dB_t$$

or, in integral form,

$$X_t = X_0 + \int_0^t f_s \, ds + \int_0^t g_s \, dB_s,$$

where $f = \{f_t : t \in [0, T]\}$ is a process integrable with respect to the Lebesgue measure with probability one and $g = \{g_t : t \in [0, T]\}$ is an Itô integrable process, and let $F : [0, T] \times \mathbb{R}$, $(t, x) \mapsto F(t, x)$ be a $\mathcal{C}^{1,2}$ function (continuously differentiable with respect to the first argument and twice continuously differentiable with respect to the second one). Then, if $Y_t = F(t, X_t)$,

$$dY_t = \frac{\partial F}{\partial t}(t, X_t) \, dt + \frac{\partial F}{\partial x}(t, X_t) \, dX_t + \frac{1}{2} \frac{\partial^2 F}{\partial x^2}(t, X_t)(dX_t)^2$$

or, in integral form,

$$Y_t = F(0, X_0) + \int_0^t \frac{\partial F}{\partial s}(s, X_s) \, ds + \int_0^t \frac{\partial F}{\partial x}(s, X_s) \, dX_s + \frac{1}{2} \int_0^t \frac{\partial^2 F}{\partial x^2}(s, X_s)(dX_s)^2.$$

In the previous theorem, the differentials dX_t and $(dX_t)^2$ can be treated as if they were finite real quantities by using the rules $dt \cdot dt = dt \cdot dB_t = dB_t \cdot dt = 0$ and $(dB_t)^2 = dt$. Then, for instance, we have that

$$\int_0^t \frac{\partial F}{\partial x}(s, X_s) \, dX_s = \int_0^t \frac{\partial F}{\partial x}(s, X_s) f_s \, ds + \int_0^t \frac{\partial F}{\partial x}(s, X_s) g_s \, dB_s,$$

and

$$\int_0^t \frac{\partial^2 F}{\partial x^2}(s, X_s)(dX_s)^2 = \int_0^t \frac{\partial^2 F}{\partial x^2}(s, X_s) g_s^2 \, ds.$$

With this, we can now begin to study stochastic differential equations.

3.2 An existence and uniqueness result

The first thing one must check when one studies equations like (3) (or its differential form), is to make sure that there is at least one solution and, if possible, to see that it is unique. It turns out that, under similar hypothesis to the ones used in Picard's Theorem on the processes g and f, one can show that there is a unique stochastic process X satisfying equation (3). However, the uniqueness is understood in the sense that any other process satisfying the SDE is a modification of our solution.

More particularly, we require f and g to be Lipschitz functions and of linear growth with respect to the second variable for each $t \in [0, T]$:

$$|f(t,x) - f(t,y)| + |g(t,x) - g(t,y)| \leq C|x-y|, \quad |f(t,x)| + |g(t,x)| \leq D(1+|x|),$$

for some positive constants C and D, and the initial condition X_0 to be deterministic (this last hypothesis can be relaxed by considering any square integrable initial condition satisfying some measurability properties). For a precise statement of the result and a proof, we refer to [4, Theorem 5.2.1].

With all this, we can finally study equations like the Langevin one, equation (1), when the noise $\dot{F}_t$ is identified with the differential of the Brownian motion. In the following section we treat a particular case of such equations and compute some observable quantities.

3.3 The case of the Langevin equation

Let us consider equation (1) when $\dot{F}_t\, dt = \sigma\, dB_t$ for some real constant σ. That is, we consider the equation

$$d\dot{X}_t = -\mu \dot{X}_t\, dt + \sigma\, dB_t,$$

for some positive constant μ and some real constant σ. The theorem of existence and uniqueness of solutions tells us that, for each $T \geq 0$ and any deterministic initial condition $\dot{X}_0$, there is a unique process (modulo modifications) $\dot{X} = \{\dot{X}_t : t \in [0, T]\}$ satisfying the above equation. To give an explicit formula for $\dot{X}_t$, we multiply the SDE by the integrating factor $e^{\mu t}$, which leads to

$$e^{\mu t}\, d\dot{X}_t + \mu e^{\mu t} \dot{X}_t\, dt = e^{\mu t} \sigma\, dB_t.$$

The usual product rule would tell us that the left-hand side can be identified with $d(e^{\mu t}\dot{X}_t)$. However, this might not be true in the context of stochastic processes. To make sure that this holds, we apply Itô's formula to the function $F(t,x) = xe^{\mu t}$, for which we have

$$\frac{\partial F}{\partial t}(t,x) = \mu x e^{\mu t}, \quad \frac{\partial F}{\partial x}(t,x) = e^{\mu t}, \quad \frac{\partial^2 F}{\partial x^2}(t,x) = 0.$$

So, indeed, we have that

$$d(e^{\mu t}\dot{X}_t) = e^{\mu t}\, d\dot{X}_t + \mu e^{\mu t}\dot{X}_t\, dt = e^{\mu t}\sigma\, dB_t$$

or, in integral form,

$$\dot{X}_t e^{\mu t} = \dot{X}_0 + \sigma \int_0^t e^{\mu s}\, dB_s.$$

Which simplifies to

$$\dot{X}_t = \dot{X}_0 e^{-\mu t} + \sigma \int_0^t e^{-\mu(t-s)}\, dB_s.$$

With this and other results like the isometry formula, we can compute some observable quantities like the mean, the variance and the covariance. A straightforward computation using the isometry formula shows that

$$\mathbb{E}[\dot{X}_t] = \dot{X}_0 e^{-\mu t}, \quad \mathbb{E}[\dot{X}_t^2] = \dot{X}_0^2 e^{-2\mu t} + \frac{\sigma^2}{2\mu}(1 - e^{-2\mu t}), \quad \mathrm{Var}(\dot{X}_t) = \frac{\sigma^2}{2\mu}(1 - e^{-2\mu t}). \tag{6}$$

Finally, for $0 \leq s < t$, we have, by letting $I_t = \int_0^t e^{\mu u}\, dB_u$ (observe that the stochastic integral involved is the one of a deterministic function, so we are under the hypothesis of Theorem 3.2),

$$\begin{aligned}
\mathrm{Cov}(\dot{X}_t, \dot{X}_s) &= \mathbb{E}[(\dot{X}_t - \mathbb{E}[\dot{X}_t])(\dot{X}_s - \mathbb{E}[\dot{X}_s])] \\
&= \sigma^2 e^{-\mu(t+s)} \mathbb{E}[I_t I_s] \\
&= \sigma^2 e^{-\mu(t+s)} \mathbb{E}[(I_t - I_s) I_s] + \sigma^2 e^{-\mu(t+s)} \mathbb{E}[I_s^2] \\
&= \sigma^2 e^{-\mu(t+s)} \mathbb{E}[I_t - I_s] \mathbb{E}[I_s] + \sigma^2 e^{-\mu(t+s)} \mathbb{E}[I_s^2] \\
&= \sigma^2 e^{-\mu(t+s)} \int_0^s e^{2\mu u}\, du \\
&= \frac{\sigma^2}{2\mu}(e^{\mu(s-t)} - e^{-\mu(t+s)}),
\end{aligned}$$

where we have used Theorem 3.2 and the isometry formula. Hence, for any $s, t \in [0, T]$,

$$\mathrm{Cov}(\dot{X}_t, \dot{X}_s) = \frac{\sigma^2}{2\mu}(e^{-\mu|t-s|} - e^{-\mu(t+s)}). \tag{7}$$

Moreover, Theorem 3.2 tells us that the process $\dot{X}$ is Gaussian with mean and covariance functions given by the first term in (6) and (7), respectively, and that, for each $t \in [0, T]$, $\dot{X}_t$ is a normal random variable with mean and variance given by the first and last terms in (6).

Acknowledgements

I would like to thank Xavier Bardina Simorra for his invaluable guidance and support and for always giving me the freedom to choose the path that suited me the most.

References

[1] P. Billingsley, *Convergence of Probability Measures*, Second edition, Wiley Ser. Probab. Statist. Probab. Statist., Wiley-Intersci. Publ., John Wiley & Sons, Inc., New York, 1999.

[2] S. Boukfal Lazaar, Weak convergence of the Lazy Random Walk to the Brownian motion, *ReportsSCM* **8(1)** (2023), 11–19.

[3] I. Karatzas, S.E. Shreve, *Brownian Motion and Stochastic Calculus*, Second edition, Grad. Texts in Math. **113**, Springer-Verlag, New York, 1991.

[4] B. Øksendal, *Stochastic Differential Equations. An Introduction with Applications*, Sixth edition, Universitext, Springer Verlag, Berlin, 2003.